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Determination of the pressure and energy density of the universe, implying the presence of missing mass

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Abstract. Pressure and energy density of the universe are computed and they are used to show that the universe must have missing mass.

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1. Introduction

The metric for a homogeneous and isotropic universe is given by the Robertson-Walker metric which is of the form given as

$$ds^2 = -dt^2 + [\lambda(t)]^2\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\} \tag{1}$$

Here the function $\lambda(t)$ is called the cosmic scaling factor and k is a constant which by the suitable choice of the coordinate r can be set to have the values 1, 0, -1. r, θ and ϕ are the co-ordinates in which the typically outmoving galaxies are at rest.

When $k = 1$, we have a closed and spherical universe which is finite but unbounded. The parameter $\lambda(t)$ can be taken as the radius of the universe. The proper circumference and proper volume of the universe is given as

$$L = 2\pi\lambda(t)$$

and

$$V = 2\pi^2\lambda^3(t)$$

Now, for $k = 0$ and $k = -1$, the universe is not closed. For $k = 0$ it is critical and for $k = -1$ it is open. $\lambda(t)$ cannot be considered as the radius of the universe, but still it sets the scale of the geometry of space and time. Therefore it is called cosmic scaling factor.

When $k = 1$ is put in the equation (1), and taking $r = sin\psi$, the spatial part of the metric gets reduced to the metric on a 3-D sphere. When $k = 0$, the spatial part of the metric gets reduced to a flat space metric. And when $k = -1$, putting $r = sinh\psi$, we get the spatial part of the metric as the metric on a hyperboloid.

Now for the Robertson-Walker metric,

$$g_{tt} = -1, \ g_{it}, \text{and} g_{kj} = \lambda^2(t)\tilde{g}_{ij}.$$

The only non-vanishing elements of the Christoffel symbol for this metric are:

$$\Gamma^t_{ij} = \lambda\dot{\lambda}\tilde{g}_{ij}$$

$$\Gamma^i_{tj} = \frac{\dot{\lambda}}{\lambda}\delta^i_j$$

and

$$\Gamma^i_{jk} = \frac{1}{2}(\tilde{g}^{-1})^{il}(\frac{\partial \tilde{g}_{ij}}{\partial x^k} + \frac{\partial \tilde{g}_{ik}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}) == \tilde{\Gamma}^i_{jk}$$

The Ricci tensor has the elements,

$$R_{tt} = \frac{3\ddot{\lambda}}{\lambda}$$

$$R_{ti} = 0$$

$$R_{ij} = \tilde{R}_{ij} - (\lambda\ddot{\lambda} + 2\dot{\lambda}^2)\tilde{g}_{ij}$$

where, $\tilde{R}_{ij}$ is the spatial Ricci tensor calculated from the metric $\tilde{g}_{ij}$,

$$\tilde{g}_{ij} = \frac{\partial \tilde{\Gamma}^k_{ki}}{\partial x^i} - \frac{\partial \tilde{\Gamma}^k_{ij}}{\partial x^k} + \tilde{\Gamma}^k_{li}\tilde{\Gamma}^i_{kj} - \tilde{\Gamma}^k_{ij}\tilde{\Gamma}^l_{kl}$$

Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}$$

which for a given matter distribution on the RHF gives the curvature and geometry of the space on the LHS.

The energy momentum tensor $T_{\mu\nu}$ which states the average state of cosmic matter must have its different components as

$$T_{00} = \rho(t)$$

is the energy density

$$T_{i0} = 0$$

is a form invariant vector field which is zero in inhomogeneous and isotropic space, and,

$$T_{ij} = g_{ij}p(t)$$

where, $p(t)$ is the pressure.

All these can be put in a single form as,

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$

which has the form of energy momentum tensor of a perfect fluid.

A perfect fluid is a fluid in which at every point we can associate a velocity vector $\vec{v}$ and an observer moving with the velocity sees the whole fluid around him to be isotropic.

U_μ is the local value of $\frac{dx^\mu}{d\tau}$ for a co-moving fluid element. p and ρ are always defined as the pressure and energy density measured by an observer in a locally inertial frame that happens to be moving with the fluid at the instant of measurement.

Now the (0,0) component of the Einstein's equation for the Robertson-Walker metric is

$$3\ddot{\lambda} = -4\pi G(\rho + 3p)\lambda \tag{i}$$

while the (i,j) component is

$$\lambda\ddot{\lambda} + 2\dot{\lambda}^2 + 2k = 4\pi G(\rho - p)\lambda^2 \tag{ii}$$

By eliminating $\ddot{\lambda}$ from equation (i) and (ii) we get

$$\dot{\lambda}^2 + k = \frac{8\pi G}{3}\rho\lambda^2 \tag{iii}$$

And from equation (ii) and (iii) we get the equation of energy conservation as

$$\frac{d}{dt}(\rho\lambda^3) = -3p\dot{\lambda}\lambda^2$$

Now, if the energy density is dominated by relativistic particles such as photons $p = \rho/3$ and we get,

$$\rho \propto \lambda^{-4}$$

whereas if the energy density is dominated by non-relativistic particles with negligible pressure,

$$\rho \propto \lambda^{-3}.$$

Here some statements are in order regarding how far one can use the special relativistic mechanics in studying the cosmological dynamics. There is

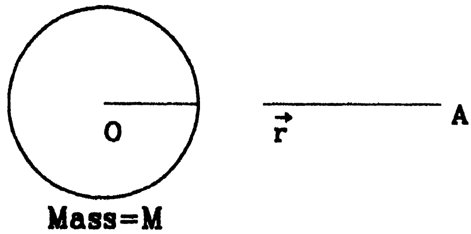

Figure 1. Cartoon diagram of a spherically symmetric body and point A in question.

a theorem called Birkoff theorem which states that a spherically symmetric gravitational field in empty space must be static, with a metric given by Schwarzschild metric. The Schwarzschild metric is given as,

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (2)$$

Now, when we calculate the gravitational field due to a spherically symmetric body at a distance r from the centre as shown in Fig. 1, we consider as if the whole mass of the body is concentrated at the centre and we get a spherically symmetric gravitational field at the point A. If the body S is expanding or collapsing with time, it produces non-static gravitational field within the body. But at point A, the expansion or contraction produces no effect since at the points outside the body we Calcutta the field by assuming as if the whole mass is concentrated at the center. So whether the body is collapsing or expanding produces no effect at points such as A. Thus the field there is static one. In fact, the gravitational radiation is produced by the change of quadrupole moment of the body. Now for a spherically symmetric body the quadrupole moment is zero. And so there is no gravitational radiation.

Suppose we want to study some physical system s, such as the solar system whose size is much less than the cosmic scale factor λ. We can imagine s to be placed in a spherical cavity cut out of the expanding universe, and so long as the size of the cavity is much less than λ we can safely consider this cavity to be empty apart from the system s. If s were absent, the gravitational field inside the cavity would be a spherically symmetric field with $R_{\mu\nu} = 0$ and hence according to Birkoff theorem it would have a flat space metric equivalent to Minkowski metric $\eta_{\mu\nu}$. As long as the system s is not too big, we can then calculate its gravitational field as a perturbation on $\eta_{\mu\nu}$, ignoring all matter

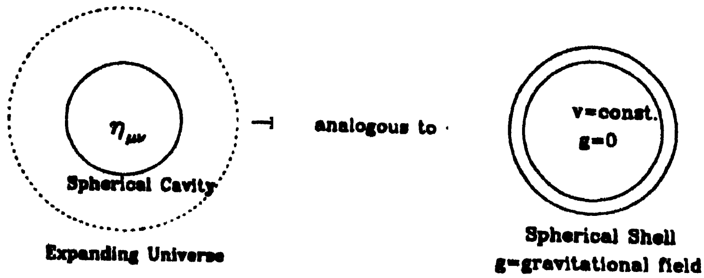

Figure 2. Analogy between the expanding universe and a spherical shell.

outside our cavity and we can determine the behaviour of the system by using Newtonian or special relativistic mechanics.

Here we are considering the application of Birkoff theorem to the gravitational field inside an empty spherical cavity at the centre of the spherically symmetric body. In this case, the metric is again given by the Schwarzschild solution. But since here the point $r = 0$ is in empty space, there can be no singularity, so the integration constant in the Schwarzschild metric vanishes and we get a flat space metric given as,

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

This is analogous to another famous result of Newtonian theory that gravitational field of a spherical shell vanishes inside the shell (Fig. 2).

To calculate the pressure and energy density of the present universe, we take equations (i) and (ii),

$$3\ddot{\lambda} = -4\pi G(\rho + 3p)\lambda \tag{i}$$

$$\lambda\ddot{\lambda} + 2\dot{\lambda}^2 + 2k = 4\pi G(\rho - p)\lambda^2 \tag{ii}$$

Putting the value of $\ddot{\lambda}$ from (i) and (ii), we get,

$$\rho = \left(\frac{k}{\lambda^2} + \frac{\dot{\lambda}^2}{\lambda^2}\right)\frac{3}{8\pi G}$$

$$\rho_0 = \left(\frac{k}{\lambda_0^2} + H_0^2\right)\frac{3}{8\pi G} \tag{3}$$

where, $H_0^2 = \frac{\dot{\lambda}^2}{\lambda^2}$

Thus, from equation (ii),

$$-q_0\dot{\lambda}^2 + 2\dot{\lambda}^2 + 2k = 4\pi G(\rho - p)\lambda^2$$

where,

$$q_0 = -\frac{\lambda\ddot{\lambda}}{\dot{\lambda}^2}.$$

Thus, the expression for pressure becomes,

$$p_0 = -\frac{1}{8\pi G}[\frac{k}{\lambda^2} + H_0^2(1 - 2q_0)] \tag{4}$$

Equation (3) and (4) represents the present energy density and pressure of the universe.

Equation (3) can be written as,

$$\frac{k}{{\lambda_0}^2}(\rho_0 - \frac{3H_0^2}{8\pi G}$$

From here it can be seen that the spatial curvature $\frac{k}{\lambda^2}$ is +ve or -ve depending on whether the present density ρ_0 of the universe is greater or less than a critical density

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{5}$$

Taking $H_0 \simeq 75$km/sec/Mpc we get the value of the critical density as

$$\rho_c = 1.1 \times 10^{-29} \ \text{gmcm}^{-3}$$

The energy density of the present universe is dominated by nonrelativistic matter and hence $p_0 << \rho_0$. Taking $p_0 \to 0$ in eqn (4) we get,

$$\frac{k}{\lambda^2} = (2q_0 - 1)H_0^2$$

Using this expression of k/λ^2 in eqn. (3) and then dividing by eqn. (5) we get,

$$\frac{\rho_0}{\rho_c} = 2q_0 \tag{6}$$

For $q_0 > \frac{1}{2}$ we get a positively curved universe with $\rho_0 > \rho_c$ and for $q_0 < \frac{1}{2}$ we get a negatively curved universe with $\rho_0 < \rho_c$. A plot of red-shift and apparent magnitude of 42 first ranked cluster galaxies is similar to Fig. 3.

From Fig. 3 it is found that credence to the value $q_0 = 1$ is to be given. Taking $H_0 = 75$km/sec/Mpc, the present density of the universe is about,

$$\rho_0 = 2\rho_c \simeq 2 \times 10^{-29}\text{gm/cm}$$

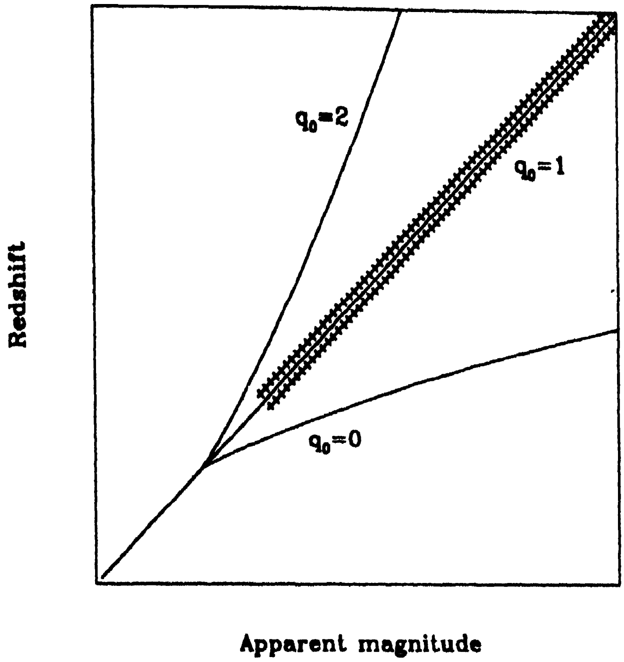

Figure 3. Schematic diagram showing red-shift vs. apparent magnitude for a few first ranked cluster galaxies ('+' signs).

Unfortunately, this result does not agree with the observed density of galactic matter. The measured galactic mass density of the universe is of the order of $\rho_G = 3.1 \times 10^{-31}$ gm/cm^3. This yields

$$\rho_G/\rho_c \simeq 0.028$$

Thus if the mass of the universe is primarily concentrated in galaxies, then $q_0 = 0.014$ and we get a negatively curved universe. But this known to be incorrect. This implies that the most of the mass of the universe is not visible to us. The galaxies visible to us do not show the whole matter content of the universe. Thus there must be some invisible mass of the universe which should compensate for the required energy density. This invisible mass is called the dark matter.

References

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